Modes in Radial Wave Beam Resonators

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Abstract—Resonant modes in ring shaped resonators formed by a reflector strip bent into a circle are discussed here. These modes are derived by superimposing two radially propagating wave beams, one converging toward the resonator axis, the other one diverging from this axis. The condition for resonance leads to an integral equation whose eigenfunctions describe the field distribution at the reflector strip and whose eigenvalues determine the diffraction loss due to the fraction of energy bypassing the reflector.

The approximations made in deriving the radial beam mode system are equivalent to those used for the derivation of the axial beam mode system in a Fabry-Perot resonator. Within the limits of these approximations the kernel of the integral equation for a ring resonator is of the same form as the kernel of the integral equation for a parallel strip resonator. If in the radial case the reflector strip is also curved within the axial planes with a radius of curvature equal to the diameter of the reflector ring, and if the width of the reflector strip is sufficiently large, the axial field distribution of the modes is described by Gauss-Hermite functions.

The Q of the ring resonator is determined by the diffraction loss and by reflection loss caused by the finite conductivity of the reflector. Formulas for the corresponding Q-values are derived. A numerical evaluation shows that in the microwave region Q-values of the order of 10^6 are feasible.

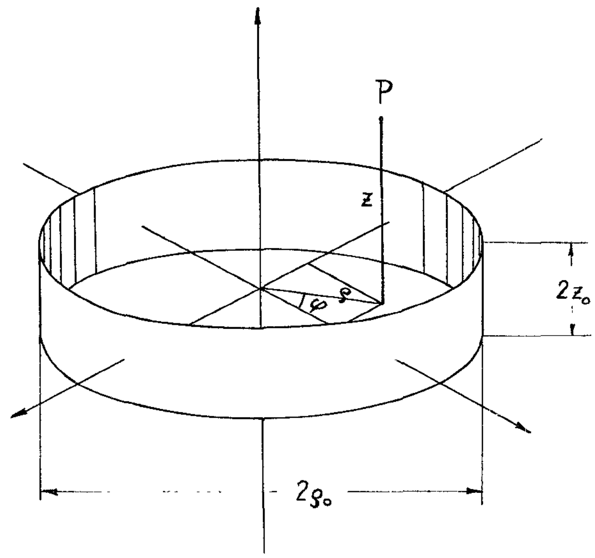

Fig. 1. Ring-shaped resonator (straight contour in planes ϕ = const.)

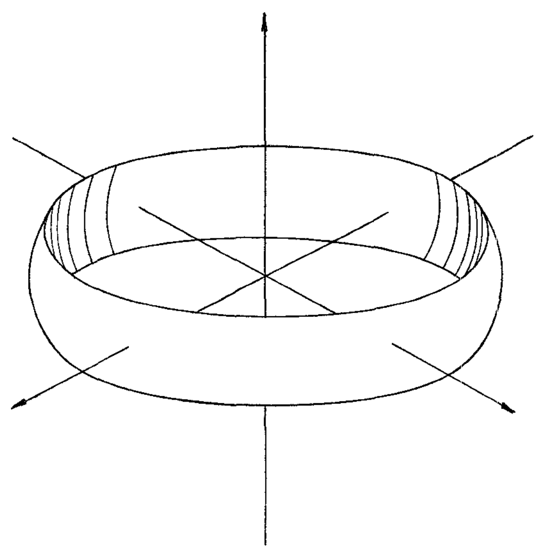

Fig. 2. Ring-shaped resonator (curved contour in planes ϕ = const.)

Introduction

The electromagnetic fields in beam waveguides and Fabry-Perot resonators can be described by a system of axially propagating beam modes whose cross-sectional field distribution is iterated at periodic intervals. In the resonator case, the period of iteration is one round trip of the beam between the two reflectors. The iteration is accomplished either by diffraction effected by limiting the beam cross section or by transformation of the cross-sectional phase distribution of the beam. The first case applies to the iris-type beam waveguides [1] and to Fabry-Perot resonators [2] with plane reflectors; the second case to lens-type beam waveguides [3], [4] and to resonators with spherical reflectors [2], [5].

This paper is concerned with ring-shaped resonators as shown in Figs. 1 and 2. The field in these resonators can be described by a system of radially propagating beam modes which have common features with the axially propagating modes in Fabry-Perot resonators.

The derivation of this radial mode system is based on the following physical consideration. Assume a beam which originates at the inner surface of the circular reflector strip and converges toward the axis of the resonator. After crossing the center area the beam diverges and returns to the reflector. The condition for resonance is that the field of the returning beam when reflected at the circular strip has the same cross-sectional amplitude and phase distribution as the original beam. Then the reflected beam can be identified with the original beam, and the assumed state of excitation is sustained. This consideration of course, disregards diffraction loss which is caused by the fact that a fraction of energy of the returning beam bypasses the reflector. This loss can be thought of as compensated for by a distributed power source at the reflector.

The mathematical formulation of the resonant condition leads to an integral equation whose eigenfunctions

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describe the fields of the resonant modes at the reflector and whose eigenvalues determine the diffraction loss. The integral equation is of the same kind as that for parallel strip resonators.

Mathematical Formulation of the Resonance Condition

The field in the resonator from the vector potentials $\Phi\bar{e}$ and $\psi\bar{e}$ where Φ and ψ satisfy the scalar wave equation. The corresponding field components are:

$$E_\rho = \frac{\partial^2\Phi}{\partial\rho\partial z} - j\frac{k}{\rho}\frac{\partial\Psi}{\partial\phi}, \qquad E_\phi = \frac{1}{\rho}\frac{\partial^2\Phi}{\partial\phi\partial z} + jk\frac{\partial\Psi}{\partial\rho},$$

$$E_z = k^2\Phi + \frac{\partial^2\Phi}{\partial z^2} \tag{1}$$

$$\sqrt{\frac{\mu}{\epsilon}}H_\rho = \frac{\partial^2\Psi}{\partial\rho\partial z} + j\frac{k}{\rho}\frac{\partial\Phi}{\partial\phi},$$

$$\sqrt{\frac{\mu}{\epsilon}}H_\phi = \frac{1}{\rho}\frac{\partial^2\Psi}{\partial\phi\partial z} - jk\frac{\partial\Phi}{\partial\rho},$$

$$\sqrt{\frac{\mu}{\epsilon}}H_z = k^2\Psi + \frac{\partial^2\Psi}{\partial z^2}. \tag{2}$$

$\bar{e}$ is the unit vector in the z-direction of a cylindrical coordinate system ρ, ϕ, z (see Fig. 1). The general solution of Φ and ψ can be written in the form

$$\Phi(\rho, \phi, z) = \sum_{m=-\infty}^{+\infty} [\Phi_m^{(1)}(\rho, z) + \Phi_m^{(2)}(\rho, z)]e^{jm\phi}$$

$$\Phi_m^{(1,2)}(\rho, z) = \int_{-\infty}^{+\infty} f_m^{(1,2)}(h)H_m^{(1,2)}(\gamma\rho)e^{-jhz}dh \tag{3}$$

$$\Psi(\rho, \phi, z) = \sum_{m=-\infty}^{+\infty} [\Psi_m^{(1)}(\rho, z) + \Psi_m^{(2)}(\rho, z)]e^{jm\phi}$$

$$\Psi_m^{(1,2)}(\rho, z) = \int_{-\infty}^{+\infty} g_m^{(1,2)}(h)H_m^{(1,2)}(\gamma\rho)e^{-jhz}dh \tag{4}$$

with

$$\gamma = \begin{cases} +\sqrt{k^2 - h^2} & \text{for } h^2 \leq k^2 \\ -j\sqrt{h^2 - k^2} & \text{for } h^2 > k^2. \end{cases} \tag{5}$$

$f_m(h)$ and $g_m(h)$ are the amplitude functions of the E- and H-waves, respectively. Assuming the time factor $e^{j\omega t}$ cylindrical waves converging toward the z-axis are described by the Hankel functions $H_m^{(1)}$, and waves diverging from this axis by the Hankel functions $H_m^{(2)}$.

Small diffraction losses can be expected only if the radially propagating beams have small angles of divergence from the $z=0$ plane. We therefore assume the amplitude functions f and g to be essentially zero for $h^2 > \bar{h}^2$ where $\bar{h} \ll k$. In particular we disregard any radial evanescent waves ($h^2 > k^2$) and furthermore restrict ρ to a finite range $0 \leq \rho \leq \bar{\rho}$ in which the phase angle of the Hankel function can be approximated by

$$\arctan\frac{N_m(\gamma\rho)}{J_m(\gamma\rho)} \approx \arctan\frac{N_m(k\rho)}{I_m(k\rho)} - \frac{1}{\pi}\frac{h^2/k^2}{J_m^2(k\rho) + N_m^2(k\rho)} \cdots \tag{6}$$

(J_m and N_m are the Bessel Functions of the first and second kind). The next term in this expansion causes a phase error smaller than δ where

$$\delta = \frac{1}{4}\bar{\rho}\frac{\bar{h}^4}{k^3} \text{ radian.} \tag{7}$$

If, for instance, $\bar{\rho}$ is 100 wavelengths and $\bar{h} = 0.1k$, we obtain $\delta = 0.005\pi$. Since $h^2 \ll k^2$, γ in the argument of the absolute value of $H_m^{(1,2)}(\gamma\rho)$ can be replaced by k. Thus

$$H_m^{\binom{1}{2}}(\gamma\rho) \approx H_m^{\binom{1}{2}}(k\rho)e^{\pm j(h/k)^2\alpha_m(k\rho)} \tag{8}$$

$$\alpha_m(k\rho) = \frac{1}{\pi}[J_m(k\rho) + N_m(k\rho)]^{-1}. \tag{9}$$

With this approximation Φ_m becomes

$$\Phi_m^{\binom{1}{2}}(\rho, z) = H_m^{\binom{1}{2}}(k\rho)\int_{-\infty}^{+\infty} f_m^{\binom{1}{2}}(h)e^{\pm j(h/k)^2\alpha_m(k\rho)}e^{-jhz}dh. \tag{10}$$

Corresponding expressions are obtained for $\psi_m^{(1)}$ and $\psi_m^{(2)}$.

In formulating the condition for resonance we assume that the fields are essentially confined to the region $\rho \leq \rho_0$, $-z_0 \leq z \leq +z_0$ where $2\rho_0 (<2\bar{\rho})$ and $2z_0$ are diameter and width of the reflector strip, respectively (see Figs. 1 and 2). This assumption is justified by the results which show that the fields decrease very rapidly in the $\pm z$-directions. In accordance with this assumption we postulate for the converging beams the following potential distributions at $\rho = \rho_0$:

$$\Phi_m^{(1)}(\rho_0, z) = \begin{cases} \overline{F}_m(z) & -z_0 \leq z \leq +z_0 \\ 0 & |z| > z_0 \end{cases} \tag{11}$$

$$\frac{\partial\Psi_m^{(1)}}{\partial\rho}(\rho_0, z) = \begin{cases} \overline{G}_m(z) & -z_0 \leq z \leq +z_0 \\ 0 & |z| > z_0. \end{cases} \tag{12}$$

The conditions $\Phi_m^{(1)}(\rho_0, z) = 0$ and $(\partial\psi_m^{(1)}/\partial\rho)(\rho_0, z) = 0$ for $|z| > z_0$ ensure that no power is supplied to the resonator from the outside.

For resonators of the kind shown in Fig. 2 which are curved within the axial planes $\phi = \text{const}$, the potential distribution functions $\overline{F}_m$ and $\overline{G}_m$ at $\rho = \rho_0$ are connected with those at the reflector surface F_m and G_m by the relations

$$\overline{F}_m(z) = F_m(z)e^{j(k/2R)z^2} \tag{13}$$

$$\overline{G}_m(z) = G_m(z)e^{j(k/2R)z^2} \tag{14}$$

where R is the radius of curvature within the planes $\phi = \text{const}$.

The amplitude functions $f_m^{(1)}$ and $g_m^{(1)}$ of the converging beams can be obtained from (3) and (4), respectively, by a Fourier transformation in z.

$$f_m^{(1)}(h) = \frac{1}{2\pi}\frac{1}{H_m^{(1)}(\gamma\rho)}\int_{-z_0}^{+z_0} F_m(\zeta)e^{+(k/2R)\zeta^2}e^{+jh\zeta}d\zeta \quad (15)$$

$$g_m^{(1)}(h) = \frac{1}{2\pi}\cdot\frac{1}{\gamma H_m^{(1)\prime}(\gamma\rho)}\int_{-z_0}^{+z_0} G_m(\zeta)e^{+j(k/2R)\zeta^2}e^{+jh\zeta}d\zeta. \quad (16)$$

At the center of the resonator the converging beam transforms into the diverging beam. The corresponding amplitude functions $f_m^{(2)}$ and $g_m^{(2)}$ are obtained from the condition that the total fields $\Phi_m^{(1)}+\Phi_m^{(2)}$ and $\Psi_m^{(1)}+\Psi_m^{(2)}$ are finite at $\rho=0$. This requires

$$f_m^{(1)}(h) = f_m^{(2)}(h), \qquad g_m^{(1)}(h) = g_m^{(2)}(h). \quad (17)$$

The potentials $\Phi_m^{(2)}$ and $\Psi_m^{(2)}$ of the diverging beam are obtained by inserting (15), (16), and (17) into (3) and (4). At $\rho=\rho_0$

$$\Phi_m^{(2)}(\rho_0, z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-z_0}^{+z_0} F_m(\zeta)e^{+j(k/2R)\zeta^2}\frac{H_m^{(2)}(\gamma\rho_0)}{H_m^{(1)}(\gamma\rho_0)} \cdot e^{-jh(z-\zeta)}d\zeta dh \quad (18)$$

$$\Psi_m^{(2)}(\rho_0, z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-z_0}^{+z_0} G_m(\zeta)e^{+j(k/2R)\zeta^2}\frac{H_m^{(2)}(\gamma\rho_0)}{\gamma H_m^{(1)\prime}(\gamma\rho_0)} \cdot e^{-jh(z-\zeta)}d\zeta dh. \quad (19)$$

The boundary condition at the reflector surface requires that the tangential electric field components be zero. Since part of the diverging beam bypasses the reflector, resonance can be sustained only if power is fed into the system, for instance by a distributed source at the reflector surface which raises the amplitude of the diverging beam by a factor p. This factor p is assumed to be positive real so that the source delivers only real power and does not detune the resonator.

In the case of resonators of the kind in Fig. 2 we again have to consider the phase shift between the reflector surface and the cylinder surface.

The resonance condition can then be formulated as follows:

$E_\phi = 0$:

$$\frac{m}{\rho_0}\frac{\partial}{\partial z}\left[\Phi_m^{(1)}(\rho_0, z)e^{-j(k/2R)z^2} + p\Phi_m^{(2)}(\rho_0, z)e^{+j(k/2R)z^2}\right] + k\frac{\partial}{\partial\rho}\left[\Psi_m^{(1)}(\rho, z)e^{-j(k/2R)z^2} + p\Psi_m^{(2)}(\rho, z)e^{+j(k/2R)z^2}\right]_{\rho=\rho_0} = 0 \quad (20)$$

$E_z = 0$:

$$\left(k^2 + \frac{\partial^2}{\partial z^2}\right)\left[\Phi_m^{(1)}(\rho_0, z)e^{-j(k/2R)z^2} + p\Phi_m^{(2)}(\rho_0, z)e^{+j(k/2R)z^2}\right] = 0 \quad \text{for } -z_0 \le z \le +z_0. \quad (21)$$

The condition $E_Z=0$ is only an approximation for resonators shown in Fig. 2, since E_Z differs somewhat from the actual tangential component in the $\phi=$const. planes. It is readily seen that the Φ- and Ψ-fields are decoupled. From (21) we find

$$\Phi_m^{(1)}(\rho_0, z)e^{-j(k/2R)z^2} + p\Phi_m^{(2)}(\rho_0, z)e^{+j(k/2R)z^2} = Ae^{\pm jkz} = 0 \qquad -z_0 \le z \le +z_0. \quad (22)$$

The constant A must be zero, since the supposition of radial beams excludes plane waves in the $\pm z$-directions. Therefore with (20)

$$\frac{\partial}{\partial\rho}\left[\Psi_m^{(1)}(\rho, z)e^{-j(k/2R)z^2} + p\Psi_m^{(2)}(\rho, z)e^{+j(k/2R)z^2}\right]_{\rho=\rho_0} = 0 \qquad -z_0 \le z \le +z_0. \quad (23)$$

Substituting in (22) $\Phi_m^{(1)}$ with (11) and (13), $\Phi_m^{(2)}$ with (18) we obtain a homogeneous integral equation of the second kind for the distribution function F_m of the electric vector potential on the reflector surface

$$F_m(z) = p\int_{-z_0}^{+z_0} K_\Phi(z, \zeta)F_m(\zeta)d\zeta \qquad -z_0 \le z \le +z_0 \quad (24)$$

$$K_\Phi(z\ \zeta) = -\frac{1}{2\pi}e^{+j(k/2R)(z^2+\zeta^2)}\int_{-\infty}^{+\infty}\frac{H_m^{(2)}(\gamma\rho_0)}{H_m^{(1)}(\gamma\rho_0)} \cdot e^{-jh(z-\zeta)}dh. \quad (25)$$

Similarly by substituting (12), (14), and (19) into (23) we obtain the integral equation for the distribution function G_m of the magnetic vector potential on the reflector surface:

$$G_m(z) = p\int_{-z_0}^{+z_0} K_\Psi(z, \zeta)G_m(\zeta)d\zeta \qquad -z_0 \le z \le +z_0 \quad (26)$$

$$K_\Psi(z, \zeta) = -\frac{1}{2\pi}e^{+j(k/2R)(z^2+\zeta^2)}\int_{-\infty}^{+\infty}\frac{H_m^{(2)\prime}(\gamma\rho_0)}{H_m^{(1)\prime}(\gamma\rho_0)} \cdot e^{-jh(z-\zeta)}dh. \quad (27)$$

Using the approximation (8) for the Hankel functions the integral in the kernel (25) can be evaluated analytically. One obtains

$$K_\Phi(z, \zeta) = \frac{1}{\sqrt{8\pi}}\frac{k}{\sqrt{\alpha_m(k\rho_0)}}e^{-j[2\beta_m(k\rho_0)+3\pi/4]} \cdot e^{j[(k/2R)(z^2+\zeta^2)-(k^2/\gamma\alpha_m(k\rho_0))(z-\zeta)^2]} \quad (28)$$

with

$$\beta_m(k\rho_0) = \arctan\frac{N_m(k\rho_0)}{J_m(k\rho_0)}\cdot$$

The kernel (27) contains the differentiated Hankel functions which can be approximated by

$$H_m^{(1)}{}'_{(2)}(\gamma\rho) = H_m^{(1)}{}'_{(2)}(k\rho)e^{\mp j(h^2/k^2)\alpha_m(k\rho)} \qquad 0 \leq \rho \leq \bar{\rho} \tag{29}$$

since amplitude terms of the order h^2/k^2 can be neglected within the range of the approximations.

Thus evaluating the integral in (27)

$$K_\Psi(z, \zeta) = \frac{1}{\sqrt{8\pi}} \cdot \frac{k}{\sqrt{\alpha_m(k\rho_0)}} e^{-j[2\beta_m'(k\rho_0)+3\pi/4]} \cdot e^{j[(k/2R)(z^2+\zeta^2)-(k/8\alpha_m(k\rho_0))(z-\zeta)^2]} \tag{30}$$

with

$$\beta_m'(k\rho_0) = \arctan \frac{N_m'(k\rho_0)}{J_m'(k\rho_0)}.$$

If $k\rho_0 \gg m^2$, α_m, β_m and β_m' can be approximated:

$$\alpha_m(k\rho_0) = \frac{1}{2} k\rho_0$$

$$\beta_m(k\rho_0) = k\rho_0 - \left(m + \frac{1}{2}\right)\frac{\pi}{2} + \frac{4m^2 - 1}{8k\rho_0}$$

$$\beta_m'(k\rho_0) = k\rho_0 - \left(m - \frac{1}{2}\right)\frac{\pi}{2} + \frac{4m^2 + 3}{8k\rho_0}. \tag{31}$$

Using the substitutions

$$\kappa = \begin{cases} pe^{-j[2\beta_m(k\rho_0)+(3\pi/4)]} \\ pe^{-j[2\beta_m'(k\rho_0)+(3\pi/4)]} \end{cases} \tag{32a}$$

$$F(x) = \begin{cases} F_m(z) & \Phi - \text{fields} \\ G_m(z) & \Psi - \text{fields} \end{cases} \tag{32b}$$

$$x = \frac{1}{2} \frac{k}{\sqrt{\alpha_m(k\rho_0)}} z, \qquad \xi = \frac{1}{2} \frac{k}{\sqrt{\alpha_m(k\rho_0)}} \zeta,$$

$$x_0 = \frac{1}{2} \frac{k}{\sqrt{\alpha_m(k\rho_0)}} z_0, \qquad 1 - 4\frac{\alpha_m(k\rho_0)}{kR} = 2\sigma \tag{32c}$$

the integral equations (24) and (26) can be reduced to a normalized form

$$F(x) = \frac{\kappa}{\sqrt{2\pi}} \int_{-x_0}^{+x_0} F(\xi) e^{-j\sigma(x^2+\xi^2)} e^{jx\xi} d\xi \qquad - x_0 \leq x \leq + x_0. \tag{33}$$

The eigenfunctions $F(x) = F^{(n)}(x)$ determine the field distribution of the various modes of the resonator at the reflector surface, and the eigenvalues $\kappa = \kappa^{(n)}$ the corresponding diffraction loss.

Discussion

The integral equation (33) is of the same form as the one which is obtained for the parallel strip resonator which has been studied by a number of authors. The case of Fig. 1 where the reflector is of cylindrical shape ($R = \infty$, $\sigma = 1/2$) corresponds to a plane-parallel strip resonator [2], [6]; the other case (Fig. 2) with $R \neq \infty$ corresponds to a parallel strip resonator with two identical parabolic reflectors [7]–[9]. If R is finite but greater than ρ_0, the field is primarily determined by the radii ρ_0, and R and depends little on the width of the reflector strip, provided

$$x_0 = \frac{1}{2} \frac{k}{\sqrt{\alpha_m(k\rho_0)}} z_0$$

is sufficiently large.

The case

$$R = 4\frac{\alpha_m(k\rho_0)}{k}$$

corresponds to the parallel strip resonator with confocal reflectors [5], [10]. If $k\rho_0 \gg 1$ the approximation for α_m in (31) applies so that $R = 2\rho_0$. The corresponding integral equation is of particular interest in that the kernel reduces to a Fourier kernel:

$$K(x, \xi) = \frac{1}{\sqrt{2\pi}} e^{jx\xi}. \tag{34}$$

The eigenfunctions of this integral equation are angular prolate spheroidal functions [11]. Using Flammer's notation [12]

$$F^{(n)}(x) = S_{on}\left(x_0^2, \frac{x}{x_0}\right). \tag{35}$$

The function S_{on} form in the range $-x_0 \leq x \leq +x_0$ a complete orthogonal system

$$\int_{-x_0}^{+x_0} S_{on}\left(x_0^2, \frac{x}{x_0}\right) S_{0k}\left(x_0^2, \frac{x}{x_0}\right) dx = \delta_{nk} N_n$$

where N_n is a normalization constant. The S_{on} are real functions of x which are even and odd with n. The corresponding eigenvalues $\kappa^{(n)}$ are real for even n and imaginary for odd n:

$$\frac{1}{\kappa^{(n)}} = j^n \sqrt{\frac{2}{\pi}} x_0 R_{on}^{(1)}(x_0^2, 1) \tag{36}$$

where $R_{on}^{(1)}$ are radial prolate spheroidal functions. In view of subsequent calculations of the Q of ring resonators, numerical values of $\kappa^{(0)}$ are given in Table I. Values for $\kappa^{(n)}$ with $n > 0$ can be found in the literature [5], [9].

TABLE I
THE LOWEST ORDER EIGENVALUE $\kappa^{(0)}$ OF INTEGRAL (33)

x_0	1.6	1.8	2.0	2.2	2.4	2.6	2.8
$\kappa^{(0)}$	1.02614	1.00808	1.00206	1.00043	1.00001	1.00008	1.00000

Equations (32a) and (36) yield relations for the resonant wavelengths $\lambda_{l,m,n}$. The quantity p in (32a) is positive real, as explained in the text following equation (19). Hence for resonant modes derived from the potential Φ we obtain

$$l+\frac{n}{4}-\frac{1}{\pi}\arctan\frac{N_m\left(2\pi\frac{\rho_0}{\lambda_{lmn}}\right)}{J_m\left(2\pi\frac{\rho_0}{\lambda_{lmn}}\right)}-\frac{3}{8}=0, \quad (37a)$$

and for the modes derived from the potential Ψ

$$l+\frac{n}{4}-\frac{1}{\pi}\arctan\frac{N_m'\left(2\pi\frac{\rho_0}{\lambda_{lmn}}\right)}{J_m'\left(2\pi\frac{\rho_0}{\lambda_{lmn}}\right)}-\frac{3}{8}=0, \quad (37b)$$

l, m and n are positive integers. It is readily seen that the modes with $l+n/4=\text{const}$ have the same resonant wavelengths. If

$$k\rho_0=2\pi\frac{\rho_0}{\lambda_{lmn}}\gg m^2$$

the approximations (31) can be used, and (37a) and (37b) are solved explicitly:

$$\lambda_{lmn}=2\rho_0\left[l+\frac{m}{2}+\frac{n}{4}-\frac{1}{8}-\frac{4m^2-1}{8\pi^2\left(l+\frac{n}{4}\right)}\right]^{-1} \quad \text{for } \Phi\text{-modes} \quad (38a)$$

$$\lambda_{lmn}=2\rho_0\left[l+\frac{m}{2}+\frac{n}{4}-\frac{5}{8}-\frac{4m^2+3}{8\pi^2\left(l+\frac{n}{4}\right)}\right]^{-1} \quad \text{for } \Psi\text{-modes.} \quad (38b)$$

The fundamental radial modes ($m=0$, $n=0$) have the resonant wavelength

$$\lambda_{l00}=2\rho_0\left[l-\frac{1}{8}+\frac{1}{8\pi^2 l}\right]^{-1} \quad \text{for } \Phi\text{-modes} \quad (39a)$$

$$\lambda_{l00}=2\rho_0\left[l-\frac{5}{8}+\frac{3}{8\pi^2 l}\right]^{-1} \quad \text{for } \Psi\text{-modes.} \quad (39b)$$

The modes whose resonant wavelengths are closest to those of the fundamental radial modes are the azimuthal modes ($l-1$, 2, 0). The difference in wavelength is:

$$\Delta\lambda=\lambda_{l-1,2,0}-\lambda_{l,0,0}=\frac{2}{\pi^2 l^2}\lambda_{l,0,0} \quad \text{for } \Phi\text{- and } \Psi\text{-modes.}$$

If, for instance, $l-40$, 100, 200, the corresponding values ρ_0/λ_{l00} are approximately 0, 50, 100, and

$$\frac{\Delta\lambda}{\lambda_{l00}}=\frac{1.25}{\pi^2}\cdot 10^{-3}, \quad \frac{2}{\pi^2}\cdot 10^{-4}, \quad \frac{5}{\pi^2}\cdot 10^{-5}, \quad (40)$$

respectively.

If the limits $\pm x_0$ of the integral in (34) increase without bound, the absolute values of the eigenvalues approach one, and the eigenfunctions can be replaced by their asymptotic representations

$$F^{(n)}(x)=\text{const. } He_n(\sqrt{2}\,x)e^{-(1/2)x^2}$$
$$=\text{const. } He_n\left(\frac{kz}{\sqrt{\alpha_m(k\rho_0)}}\right)e^{-k^2z^2/8\alpha_m(k\rho_0)}, \quad (41)$$

where He_n are Hermite polynomials [13]. For $m=n=0$ the asymptotic representation is already a good approximation if x_0 exceeds two. The eigenfunctions of the integral equation describe the fields only at the reflector surface. In order to obtain the field distribution of the resonant modes in the entire space $0\le\rho\le\rho_0$, one can employ the same method used for the derivation of the mode systems in beam waveguides [4], [3]. According to this method the amplitude functions $f_m(h)$ are expanded into a complete system of orthogonal functions. In the case of ring resonators of the type of Fig. 2 when $2\alpha_m(k\rho_0)/k<R<\infty$, the appropriate functions are Gaussian-Hermite functions. The corresponding beam mode system will be discussed in a future paper.[1]

Derivation of the Q of Ring-Resonators

The Q of a resonator is usually defined by

$$Q=\frac{\omega W}{N} \quad (42)$$

where W is the stored energy and N is the energy loss per second. The loss N consists of two parts, the diffraction loss N_d and the conductivity loss N_c, assuming that there are no dielectric losses involved. Correspondingly we can define two Q's:

$$Q_d=\frac{\omega W}{N_d} \qquad Q_c=\frac{\omega W}{N_c} \quad (43)$$

with

$$\frac{1}{Q}=\frac{1}{Q_d}+\frac{1}{Q_c}. \quad (44)$$

First, we determine the stored energy W in terms of the amplitude functions $f(h)$ and $g(h)$

$$W_e=\frac{1}{2}\epsilon\iiint \boldsymbol{E}\cdot\boldsymbol{E}^*dV \quad (45)$$

$W=W_e+W_m$ with

$$W_m=\frac{1}{2}\mu\iiint \boldsymbol{H}\cdot\boldsymbol{H}^*dV. \quad (46)$$

The integration is extended over the entire cylindrical space $\rho\le\rho_0$. The energy outside this space can be regarded negligibly small.

[1] To be presented at the URSI Symposium on Electro-magnetic Theory, Delft, Holland.

The scalar products $\boldsymbol{EE}^*$ and $\boldsymbol{HH}^*$ in (45) and (46) are expressed in terms of the potential functions Φ and Ψ. In the case of a field derived from the potential

$$\Phi(\rho, \phi, z) = \sum_{m=-\infty}^{+\infty} \Phi_m(\rho, z) e^{jm\phi} \tag{47}$$

we obtain with (1) and (2) after performing the integration over ϕ

$$W_e = \pi\epsilon \sum_{m=-\infty}^{+\infty} \int_0^{\rho_0} \int_{-\infty}^{+\infty} \left[\frac{\partial^2 \Phi_m}{\partial\rho\partial z} \frac{\partial^2 \Phi_m^*}{\partial\rho\partial z} + \frac{m^2}{\rho^2} \frac{\partial\Phi_m}{\partial z} \frac{\partial \Phi_m^*}{\partial z}, + \left(k^2\Phi_m + \frac{\partial^2\Phi_m}{\partial z^2}\right)\left(k^2\Phi_m^* + \frac{\partial^2 \Phi_m^*}{\partial z^2}\right)\right] \rho d\rho dz \tag{48}$$

$$W_m = \pi\mu \sum_{m=-\infty}^{+\infty} \int_0^{\rho_0} \int_{-\infty}^{+\infty} \left[\frac{m^2k^2}{\rho^2} \Phi_m\Phi_m^* + k^2 \frac{\partial \Phi_m}{\partial\rho} \frac{\partial\Phi_m^*}{\partial \rho}\right] \rho d\rho dz. \tag{49}$$

Since the field is continuous at $\rho=0$, the amplitude functions of the converging and the diverging beam must be equal

$$f_m^{(1)}(h) = f_m^{(2)}(h) = f_m(h). \tag{50}$$

Hence, with (3)

$$\Phi_m(\rho, z) = \Phi_m^{(1)}(\rho, z) + \Phi_m^{(2)}(\rho, z) = 2\int_{-\infty}^{+\infty} f_m(h) J_m(\gamma\rho) e^{-jhz} dh. \tag{51}$$

Equation (51) is inserted into (48) and (49). Performing the integrations in ρ and z we obtain:

$$W_e = 4\pi^2 \epsilon k^2 \rho_0^2 \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_m(h) f_m^*(h) \cdot \left| J_m'^2(\gamma\rho_0) + \left(1 - \frac{m^2}{\gamma^2\rho_0^2}\right) J_m^2(\gamma\rho_0) + \frac{h^2}{k^2} \frac{2}{\gamma\rho_0} J_m(\gamma\rho_0) J_m'(\gamma\rho_0) \right| \gamma\gamma^* dh \tag{52}$$

$$W_m = 4\pi^2 \epsilon k^2 \rho_0^2 \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_m(h) f_m^*(h) \cdot \left| J_m'^2(\gamma\rho_0) + \left(1 - \frac{m^2}{\gamma^2\rho_0^2}\right) J_m^2(\gamma\rho_0) + \frac{2}{\gamma\rho_0} J_m(\gamma\rho_0) J_m'(\gamma\rho_0) \right| \gamma\gamma^* dh. \tag{53}$$

For fields derived from the potential Ψ the expressions for W_e and W_m are interchanged. The functions f_m are replaced by the functions g_m.

The amplitude functions of the resonator fields have been assumed to be restricted to the range $|h| \leq \bar{h} \ll k$. Therefore, the integration in (52), (53) is essentially limited to the finite interval $-\bar{h} \leq h \leq +\bar{h}$. If $k\rho_0 \gg 1$ the Bessel functions in (52) and (53) can be replaced by their asymptotic representations, since for $h \leq \bar{h}$, γ is close to k. Neglecting in the integrands of (52) and (53) the higher order terms of $1/\gamma\rho_0$ we obtain

$$W_e = W_m = 8\pi\epsilon k^3 \rho_0 \Gamma \tag{54}$$

with

$$\Gamma = \begin{cases} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_m(h) f_m^*(h) dh & \text{for } \Phi\text{-modes} \\ \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} g_m(h) g_m^*(h) dh, & \text{for } \Psi\text{-modes.} \end{cases} \tag{55}$$

The diffraction loss N_d is the difference between the real powers of the converging beam and that portion of the diverging beam which is intercepted by the reflector. Since the field distribution functions of both beams are the same at the reflector except that the energy densities differ by a factor $p^2 = \kappa\kappa^*$

$$N_d = \left(1 - \frac{1}{\kappa\kappa^*}\right) N^{(1)} \tag{56}$$

where

$$N^{(1)} = \mathrm{Re}\left[\int_0^{2\pi} \int_{-\infty}^{+\infty} (E_z^{(1)} H_\phi^{(1)*} - E_\phi^{(1)} H_z^{(1)*}) \rho_0 d\phi dz\right] \tag{57}$$

is the power of the converging beam. The conductivity loss is determined by the currents on the reflector surface:

$$N_c = \frac{1}{\sigma} \int_0^{2\pi} \int_{-\infty}^{+\infty} (H_\phi H_\phi^* + H_z H_z^*) \rho_0 d\phi dz \tag{58}$$

with σ denoting the surface conductivity of the reflector. The z-integrations can be extended to the infinite range $-\infty \leq z \leq +\infty$ because the fields at $\rho = \rho_0$ are very small outside the range $-z_0 \leq z \leq +z_0$. The field components can again be expressed in terms of the amplitude functions f and g. Using the same approximations as before N_d and N_c become

$$N_d = 8\pi \sqrt{\frac{\epsilon}{\mu}} k^3 \left(1 - \frac{1}{\kappa\kappa^*}\right) \Gamma \tag{59}$$

$$N_c = 32\pi \frac{\epsilon}{\mu} \frac{k^3}{\sigma} \Gamma. \tag{60}$$

From (54), (59) and (60) we obtain

$$Q_d = \frac{2k\rho_0}{1 - \frac{1}{\kappa\kappa^*}} \qquad Q_c = \frac{1}{2} \sigma \sqrt{\frac{\mu}{\epsilon}} k\rho_0. \tag{61}$$

Since the surface conductivity decreases with $\omega^{-1/2}$, Q_c

TABLE II
Q_d OF FUNDAMENTAL RADIAL BEAM MODES (1, 0.0) FOR VARIOUS VALUES OF THE PARAMETERS ρ_0/λ AND $x_0=\sqrt{(k/2\rho_0)}\,z_0$

	$x_0=1.6$	1.8	2.0	2.2	2.4	2.6	2.8
$\rho_0/\lambda=20$	$Q_d=5.00\ 10^3$	$1.57\ 10^4$	$6.10\ 10^4$	$2.9\ 10^5$	$2\cdot10^6$	$1\cdot10^7$	$2\cdot10^7$
50	$1.25\ 10^4$	$3.93\ 10^4$	$1.53\ 10^5$	$7.3\ 10^5$	$4\cdot10^6$	$3\cdot10^7$	$5\cdot10^7$
100	$2.50\ 10^4$	$7.86\ 10^4$	$3.05\ 10^5$	$1.5\ 10^6$	$8\cdot10^6$	$6\cdot10^7$	$1\cdot10^8$

increases proportional to $\omega^{+1/2}$. As a numerical example we consider a fundamental mode with the mode numbers $m=n=0$, $l=40$, 100, and 200. The wavelength is assumed to be 0.4 cm.

From (39a) and (39b) we obtain approximately

$$\frac{\rho_0}{\lambda} = 20, 50 \text{ and } 100 \; \rho_0 = 8 \text{ cm, } 20 \text{ cm and } 40 \text{ cm.}$$

If the reflector material is copper with a surface conductivity $\sigma=9.73\omega^{-1/2}\text{ohm}^{-1}s^{-1}$.

$$Q_c = 3.36.10^5,\ 8.39.10^5, \text{ and } 1.68.10^6, \text{ respectively.} \qquad (62)$$

Q_d can be calculated using the eigenvalues $\kappa^{(0)}$ of Table I. The results are given in Table II, which shows that Q_d increases rapidly with increasing $x_0=\sqrt{(k/2\rho_0)}z_0$. For $x_0=2.2$, Q_d is of the same order in magnitude as Q_c. For $x_0=2.8$, Q_d is already two orders in magnitude greater than Q_c, and the diffraction loss is negligible compared with the conductivity loss.

In order to obtain separation of the resonances of the adjacent modes $(l, 0, 0)$ and $(l-1, 2, 0)$, the Q's must satisfy the condition

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda_{l,0,0} - \lambda_{l-1,2,0}}{\lambda_{l,0,0}} > \left(\frac{1}{Q_c} + \frac{1}{Q_d}\right)_{l,0,0}. \qquad (63)$$

Numerical values for $\Delta\lambda/\lambda$ have been given in (40). A comparison of these values with Q_c and Q_d from (62) and Table II shows that condition (63) is satisfied if x_0 is about 2 or greater.

At the 1965 G-MTT Symposium D. H. Auston and P. F. Primich informed the authors that they were studying ring-type resonators as devices for plasma diagnostics.

References

[1] J. R. Christian and G. Goubau, "Some measurements on an iris beam waveguide," *Proc. IRE (Correspondence)*, vol. 49, pp. 1679–80, November 1961.

[2] A. G. Fox and T. Li, "Resonant modes in a maser interferometer," *Bell Sys. Tech. J.*, vol. 40, pp. 453–488, March 1961.

[3] J. R. Christian and G. Goubau, "Experimental studies on a beam waveguide for millimeter waves," *IRE Trans. on Antennas and Propagation*, vol. AP-9, pp. 256–263, May 1961.

[4] G. Goubau and F. Schwering, "On the guided propagation of electromagnetic wave beams," *IRE Trans. on Antennas and Propagation*, vol. AP-9, pp. 248–256, May 1961.

[5] G. D. Boyd and J. P. Gordon, "Confocal multimode resonator for millimeter through optical wavelength masers," *Bell Sys. Tech. J.*, vol 40, pp. 489–508, March 1961.

[6] H. Schachter and L. Bergstan, "Stationary modes in optic and quasi-optic cavities," *1963 Proc. Symp. on Optical Masers*. New York: Polytechnic Press, pp. 173–198.

[7] R. F. Soohoo, "Nonconfocal multimode resonators for Masers," *Proc. IEEE* vol. 51, pp. 70–75, January 1963.

[8] W. Streifer and Hideya Gamo, "On the Schmidt expansion for optical resonator modes," *1964 Proc. Symp. on Quasi Optics*, New York: Polytechnic Press, pp. 351–365.

[9] F. Schwering, "Reiterative wavebeams of rectangular symmetry," Archiv d. Übertragung, 15, pp. 555–564, December 1961.

[10] G. D. Boyd and H. Kogelnik, "Generalized confocal resonator theory," *Bell Sys. Tech. J.*, vol. 41, pp. 1347–1369, July 1962.

[11] D. Slepian and H. O. Pollak, "Prolate spheroidal wave functions, Fourier Analysis and Uncertainty I.", *Bell Sys. Tech. J.*, vol. 40, pp. 43–63, January 1961.

[12] C. Flammer, *Spheroidal wave functions*. Stanford, Calif.: Stanford University Press, 1957,

[13] W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics*. New York: Chelsea, 1954, pp. 80–82.